The Standard Form of a Differential Equation

Format required to solve a differential equation or a system of differential equations using one of the command-line differential equation solvers such as

rkfixed, *Rkadapt*, *Radau*, *Stiffb*, *Stiffr* or *Bulstoer*.

For a numerical routine to solve a differential equation (DE), we must somehow pass the differential equation as an argument to the solver routine. A standard form for all DEs will allow us to do this.

Basic idea: get rid of any second, third, fourth, etc. derivatives that appear, leaving only first derivatives.

Example 1

$$\frac{d^2}{dx^2}y(x) + 3 \cdot \frac{d}{dx}y(x) - 5 \cdot y(x) = 4 \cdot x^5$$

This DE contains a second derivative. How do we write a second derivative as a first derivative? A second derivative is a first derivative of a first derivative.

$$\frac{d^2}{dx^2}y(x) = \frac{d}{dx}\frac{d}{dx}y(x)$$

Step 1: STANDARDIZATION

Let's define two functions $y_0(x)$ and $y_1(x)$ as $\qquad y_0(x) = y(x) \qquad$ and $\qquad y_1(x) = \frac{d}{dx}y_0(x)$

Then this differential equation can be written as $\qquad \frac{d}{dx}y_1(x) + 3 \cdot y_1(x) - 5 \cdot y_0(x) = 4x^5$

This DE contains 2 functions instead of one, but there is a strong relationship between these two functions

$$y_1(x) = \frac{d}{dx}y_0(x)$$

So, the original DE is now a system of two DEs,

$$y_1(x) = \frac{d}{dx} y_0(x) \qquad\qquad \text{and} \qquad\qquad \frac{d}{dx} y_1(x) + 3 \cdot y_1(x) - 5 \cdot y_0(x) = 4 \cdot x^5$$

The convention is to write these equations with the derivatives alone on the left-hand side.

$$\frac{d}{dx} y_0(x) = y_1(x)$$

This is the first step in the standardization process.

$$\frac{d}{dx} y_1(x) = 4 \cdot x^5 + 5 \cdot y_0(x) - 3 \cdot y_1(x)$$

Step 2: DEFINE A SINGLE FUNCTION FOR THE NUMERICAL SOLVER

$$\frac{d}{dx} y_0(x) = y_1(x) \qquad\qquad \frac{d}{dx} y_1(x) = 4 \cdot x^5 + 7 \cdot y_0(x) - 5 \cdot y_1(x)$$

We have two functions of x, $\qquad y_0(x) \qquad$ and $\qquad y_1(x)$

These 2 functions y_0 and y_1 allow us to define a single function D $\qquad D(x,y) := \begin{pmatrix} y_1 \\ 4 \cdot x^5 + 7 \cdot y_0 - 5 \cdot y_1 \end{pmatrix}$

which specifies for each vector, the corresponding vector of derivative values.

This function *D* is the essential component to be used in solving a differential equation numerically. When constructing this function, be sure to use the array subscript to indicate the order of the derivative.

use of rkfixed to solve differential equations: *rkfixed(ic, x1, x2, npoints, D)*

This function returns a matrix in which the first column contains the points at which the ODE solution is evaluated and the remaining columns contains the corresponding values of the ODE solution(s) and its first n-1 derivatives. A fourth-order <u>Runge-Kutta method</u> is used.

- *ic* is a vector of n initial values or a single initial value.
- *x1, x2* are endpoints of the interval on which the solution to differential equations will be evaluated. Initial values in y are the values at x1.
- *npoints* is the number of points beyond the initial point at which the solution is to be approximated. This controls the number of rows (1 + npoints) in the matrix returned by rkfixed.
- *D* is an n-element vector-valued function containing first derivatives of unknown functions.
When using this function you will better off by not including the units. Units can always be added after the calculations are done.

Initial Conditions $\quad\quad$ $ic := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad\quad$ $x1 := 0$ $\quad\quad$ $x2 := 5$ $\quad\quad$ npoints $:= 100$

$$Q := rkfixed(ic, x1, x2, npoints, D)$$

$$Q = \begin{array}{c|c|c}
 & 0 & 1 \\
\hline
0 & 0 & 0 \\
\hline
1 & 0.05 & 0.044 \\
\hline
2 & 0.1 & 0.08 \\
\hline
3 & 0.15 & 0.108 \\
\hline
4 & 0.2 & 0.132 \\
\hline
5 & 0.25 & 0.153 \\
\hline
6 & 0.3 & 0.172 \\
\end{array}$$
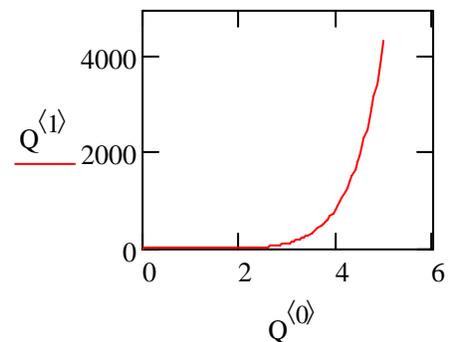
The idea is to change the n-th order ODE into a system of n coupled first-order differential equations

Systems of Differential Equations
Example 2

It may be that you are solving a system of equations rather than a single differential equation. Consider the following system

$$u''(x) = 2v(x) - x$$

$$v''(x) = 4 \cdot v(x) + 2 \cdot u(x)$$

Define

$$y_0(x) = u(x) \qquad\qquad y_1(x) = v(x)$$

The system of 2 equations becomes the system $\qquad \dfrac{d}{dx} y_0(x) = y_2(x) \qquad$ and $\qquad \dfrac{d}{dx} y_1(x) = y_3(x)$

$$\dfrac{d}{dx} y_2(x) = 2 \cdot y_1(x) - x$$

$$\dfrac{d}{dx} y_3(x) = 4 \cdot y_1(x) + 2 \cdot y_0(x)$$

then $\qquad D(x,y) := \begin{pmatrix} y_2 \\ 2 \cdot y_1 - x \\ y_3 \\ 4 \cdot y_1 + 2 \cdot y_0 \end{pmatrix} \qquad \begin{pmatrix} u'(x) \\ u''(x) \\ v'(x) \\ v''(s) \end{pmatrix}$

In this case, we have defined functions so that the even indices correspond to the unknown function u and it derivatives, while the odd indices correspond to v.

Now that the derivative vector D is set, we'll solve the second-order system using *Rkadapt*.

Rkadapt takes the following arguments:

Rkadapt(ic, start, end, npts, D)

ic - vector of initial conditions. Scalar for a first-order DE.
start, end - endpoints of the solution interval
npts - number of approximating points past the initial condition. The result will be a matrix containing npts + 1 rows
D - the derivative vector

$$\text{ic} := \begin{pmatrix} 1.5 \\ 1 \\ 1.5 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} u(0) \\ v(0) \\ u'(0) \\ v'(0) \end{pmatrix} \qquad \text{Initial condition vector}$$

$$D(x,y) := \begin{pmatrix} y_2 \\ y_3 \\ 2 \cdot y_1 - x \\ 4 \cdot y_1 + 2 \cdot y_0 \end{pmatrix} \qquad \begin{pmatrix} u'(x) \\ v'(x) \\ u''(x) = 2v(x) - x \\ v''(x) = 4 \cdot v(x) + 2 \cdot u(x) \end{pmatrix}$$

Vector-valued function containing the derivatives of the unknown functions

$$Z := \text{Rkadapt}(ic, 0, 1, 200, D)$$

Rkadapt returns a vector of data where the first column contains the values of the independent variable. The remaining columns contain the function and derivatives evaluated at the corresponding values of the independent variable. For a system, the columns are returned in the same order in which the initial conditions are entered.

	x	u	v	u'	v'

$Z =$

	0	1	2	3
0	0	1.5	1	1.5
1	$5 \cdot 10^{-3}$	1.508	1.005	1.51
2	0.01	1.515	1.01	1.52
3	0.015	1.523	1.016	1.53
4	0.02	1.53	1.021	1.54
5	0.025	1.538	1.027	1.55
6	0.03	1.546	1.033	1.561
7	0.035	1.554	1.039	1.571
8	0.04	1.562	1.046	1.581
9	0.045	1.57	1.052	1.591

$$x := Z^{\langle 0 \rangle}$$

$$u := Z^{\langle 1 \rangle}$$

$$v := Z^{\langle 2 \rangle}$$

$$u' := Z^{\langle 3 \rangle} \qquad uu := Z^{\langle 3 \rangle}$$

$$v' := Z^{\langle 4 \rangle} \qquad vv := Z^{\langle 4 \rangle}$$

- u(x)
- v(x)
- u'(x)
- v'(x)